SPRING 2025 MATH 590: QUIZ 3

Name:

1. Given three column vectors $v_1, v_2, v_3 \in \mathbb{R}^3$, how does one determine computationally that these vectors form a basis for $\mathbb{R}^3$? (3 points)

Solution. Apply elementary row operations to the matrix $A = [v_1\ v_2\ v_3]$, to put A into reduced row echelon form. If this yields the identity matrix, then then vectors form a basis for $\mathbb{R}^3$.

2. Find a basis for, and the dimension of, the space of 2×2 real symmetric matrices. You must justify your answer. (3 points)

Solution. A typical symmetric matrix is $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, showing that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span the space of symmetric matrices.

Suppose $r \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + s \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, Then $\begin{pmatrix} r & s \\ s & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, showing that $r = s = t = 0$, so the given matrices are linearly independent, and thus form a basis for the space of symmetric matrices, which therefore has dimension 3.

3. Suppose V has dimension four over $\mathbb{R}$, $V = W_1 \oplus W_2$, $S_1 = \{u, u'\}$ is a basis for W_1 and $S_2 = \{w, w'\}$ is a bases for W_2. Prove that $S_1 \cup S_2$ is a basis for V. (4 points)

Solution. Take $v \in V$. Then $v = w_1 + w_2$, for $w_1 \in W_1$ and $W_2 \in W_2$, since $V = W_1 \oplus W_2$. Now write $w_1 = au + bu'$ and $w_2 = cw + dw'$, for $a, b, c, d \in \mathbb{R}$. Then $v = au + bu' + cw + dw'$, showing that $S_1 \cup S_2$ span V.

Now suppose $ru + su' + tw + hw' = \vec{0}$, with $r, s, t, h \in \mathbb{R}$. Then $ru + su' = -tw - hw'$. The left hand side belongs to W_1 and the right hand side belongs to W_2, so this vector belongs to $W_1 \cap W_2 = \vec{0}$. Thus, $ru + su' = \vec{0}$. But u, u' are linearly independent, so $r = s = 0$. Similarly, $tw + hw' = \vec{0}$, so $t = h = 0$, since w, w' are linearly independent. Thus, $r = s = t = h = 0$, showing that u, u', w, w' are linearly independent, and hence a basis for V.